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ALGEBRAS ASSIGNED TO TERNARY RELATIONS

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Abstract. We show that to every centred ternary relation T on a set A there can be assigned (in a non-unique way) a ternary operation t on A such that the identities satisfied by $(A; t)$ reflect relational properties of T . We classify ternary operations assigned to centred ternary relations and we show how the concepts of relational subsystems and homomorphisms are connected with subalgebras and homomorphisms of the assigned algebra $(A; t)$. We show that for ternary relations having a non-void median can be derived so-called median-like algebras $(A; t)$ which become median algebras if the median $M_T(a, b, c)$ is a singleton for all $a, b, c \in A$. Finally, we introduce certain algebras assigned to cyclically ordered sets.

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In [2] and [3], the first and the third author showed that to certain relational systems $\mathcal{A} = (A; R)$, where $A \neq \emptyset$ and R is a binary relation on A , there can be assigned a certain groupoid $\mathcal{G}(A) = (A; \circ)$ which captures the properties of R . Namely, we have $x \circ y = y$ if and only if $(x, y) \in R$. In these papers we worked with so-called directed relational systems, i. e. for all $x, y \in A$ we have

$$U_R(x, y) := \{z \in A \mid (x, z), (y, z) \in R\} \neq \emptyset.$$

We are inspired by the idea of assigning a groupoid (called directoid) to a directed poset. This idea has its origin in the paper [6] by J. Ježek and R. Quackenbush. Then some structural properties of the assigned groupoid $\mathcal{G}(A)$ can be used for introducing certain structural properties of $\mathcal{A} = (A; R)$; in particular, we introduced congruences, quotient relational systems and homomorphisms which are in accordance with the corresponding concepts in $\mathcal{G}(A)$.

Hence, there arises the natural question if a similar way can be used for ternary relational systems and algebras with one ternary relation. In a particular case, such a

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correspondence exists. It is for the ternary relation “betweenness” and the so-called median algebras, see e.g. [1, 5] or [11].

However, there exist also other useful ternary relations for which a similar construction is not already derived, in particular the so-called cyclic orders, see e.g. [4, 7, 8] and [9].

Moreover, more general ternary relations were already investigated in [10] and [11] and hence our problem can be extended to a more general case than betweenness. However, to get a construction of a ternary operation, a certain restriction on the ternary relation is necessary.

In the following let A denote a fixed arbitrary non-empty set.

1. TERNARY OPERATIONS ASSIGNED TO TERNARY RELATIONS

We introduce the following concepts:

Definition 1. Let T be a ternary relation on A and $a, b \in A$. The set

$$Z_T(a, b) := \{x \in A \mid (a, x, b) \in T\}$$

is called the **centre of (a, b) with respect to T** . The ternary relation T on A is called **centred** if $Z_T(a, b) \neq \emptyset$ for all elements $a, b \in A$.

Definition 2. Let T be a ternary relation on A and $a, b, c \in A$. The set

$$M_T(a, b, c) := Z_T(a, b) \cap Z_T(b, c) \cap Z_T(c, a)$$

will be called the **median of (a, b, c) with respect to T** .

The concept of a median was originally introduced in lattices and structures derived from lattices. In particular, two sorts of medians are usually considered: $m(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ and $M(x, y, z) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$.

Now we show that to every centred ternary relation there can be assigned ternary operations.

Definition 3. Let T be a centred ternary relation on A and t a ternary operation on A satisfying

$$t(a, b, c) \begin{cases} = b & \text{if } (a, b, c) \in T \\ \in Z_T(a, c) & \text{otherwise.} \end{cases}$$

Such an operation t is called **assigned to T** .

Remark 1. By definition, if T is a centred ternary relation on A and t assigned to T then $(a, t(a, b, c), c) \in T$ for all $a, b, c \in A$.

Lemma 1. Let T be a centred ternary relation on A and t an assigned operation. Let $a, b, c \in A$. Then $(a, b, c) \in T$ if and only if $t(a, b, c) = b$.

Proof. By Definition 3, if $(a, b, c) \in T$ then $t(a, b, c) = b$. Conversely, assume $(a, b, c) \notin T$. Then $t(a, b, c) \in Z_T(a, c)$. Now $t(a, b, c) = b$ would imply $(a, b, c) = (a, t(a, b, c), c) \in T$ contradicting $(a, b, c) \notin T$. Hence $t(a, b, c) \neq b$. \square

To illuminate the role of the median, let us consider the following example:

Example 1. Let $\mathcal{L} = (L; \vee, \wedge)$ be a lattice. Define a ternary operation T on L as follows:

$$(a, b, c) \in T \quad \text{if and only if} \quad a \wedge c \leq b \leq a \vee c.$$

Put $m(x, y, z) := (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ and $M(x, y, z) := (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$. If $p \in M_T(a, b, c)$ then $p \in Z_T(a, b)$, $p \in Z_T(b, c)$ and $p \in Z_T(c, a)$, i. e. $a \wedge b \leq p \leq a \vee b$, $b \wedge c \leq p \leq b \vee c$ and $c \wedge a \leq p \leq c \vee a$ whence $m(a, b, c) \leq p \leq M(a, b, c)$. This yields

$$M_T(a, b, c) = [m(a, b, c), M(a, b, c)],$$

the interval in \mathcal{L} . It is well-known that $m(x, y, z) = M(x, y, z)$ if and only if \mathcal{L} is distributive. Hence, \mathcal{L} is distributive if and only if $|M_T(a, b, c)| = 1$ for all $a, b, c \in L$.

The previous example was used in [5] for the definition of a median algebra. If \mathcal{L} is a distributive lattice then the algebra $(L; m)$ is called the median algebra derived from \mathcal{L} . Of course, there exist median algebras which are not derived from a lattice, see [1] for details, but in every median algebra there can be introduced a ternary relation “between” by putting

$$(a, b, c) \in T_m \quad \text{if and only if} \quad m(a, b, c) = b.$$

In what follows, we show how this construction can be generalized and we get a characterization of some important properties of ternary relations by means of identities of their assigned operations.

Theorem 1. *A ternary operation t on A is assigned to some centred ternary relation T on A if and only if it satisfies the identity*

$$t(x, t(x, y, z), z) = t(x, y, z). \quad (1.1)$$

Proof. Let $a, b, c \in A$.

Assume that T is a ternary relation on A and t an assigned operation. If $(a, b, c) \in T$ then $t(a, b, c) = b$ and hence $t(a, t(a, b, c), c) = t(a, b, c)$. If $(a, b, c) \notin T$ then $t(a, b, c) \in Z_T(a, c)$ and hence $(a, t(a, b, c), c) \in T$ which yields $t(a, t(a, b, c), c) = t(a, b, c)$. Thus t satisfies identity (1.1).

Conversely, assume $t : A^3 \rightarrow A$ satisfies (1.1) and define $T := \{(x, y, z) \in A^3 \mid t(x, y, z) = y\}$. If $(a, b, c) \in T$ then $t(a, b, c) = b$ and, if $(a, b, c) \notin T$ then $(a, t(a, b, c), c) \in T$ whence $t(a, b, c) \in Z_T(a, c)$, i. e. t is assigned to T . \square

We can consider a number of properties of ternary relations which are used in [1–11] for “betweenness” and for “cyclic orders”.

Definition 4. Let T be a ternary relation on A . We call T

- **reflexive** if $|\{a, b, c\}| \leq 2$ implies $(a, b, c) \in T$;
- **symmetric** if $(a, b, c) \in T$ implies $(c, b, a) \in T$;

- **antisymmetric** if $(a, b, a) \in T$ implies $a = b$;
- **cyclic** if $(a, b, c) \in T$ implies $(b, c, a) \in T$;
- **R -transitive** if $(a, b, c), (b, d, e) \in T$ implies $(a, d, e) \in T$;
- **t_1 -transitive** if $(a, b, c), (a, d, b) \in T$ implies $(d, b, c) \in T$;
- **t_2 -transitive** if $(a, b, c), (a, d, b) \in T$ implies $(a, d, c) \in T$;
- **R -symmetric** if $(a, b, c) \in T$ implies $(b, a, c) \in T$;
- **R -antisymmetric** if $(a, b, c), (b, a, c) \in T$ implies $a = b$;
- **non-sharp** if $(a, a, b) \in T$ for all $a, b \in A$;
- **cyclically transitive** if $(a, b, c), (a, c, d) \in T$ implies $(a, b, d) \in T$.

Theorem 2. *Let T be a centred ternary relation on A and t an assigned operation. Then (i) – (xi) hold:*

(i) *T is reflexive if and only if t satisfies the identities*

$$t(x, x, y) = t(y, x, x) = t(y, x, y) = x.$$

(ii) *T is symmetric if and only if t satisfies the identity*

$$t(z, t(x, y, z), x) = t(x, y, z).$$

(iii) *T is antisymmetric if and only if t satisfies the identity*

$$t(x, y, x) = x.$$

(iv) *T is cyclic if and only if t satisfies the identity*

$$t(t(x, y, z), z, x) = z.$$

(v) *T is R -transitive if and only if t satisfies the identity*

$$t(x, t(t(x, y, z), u, v), v) = t(t(x, y, z), u, v).$$

(vi) *T is t_1 -transitive if and only if t satisfies the identity*

$$t(t(x, u, t(x, y, z)), t(x, y, z), z) = t(x, y, z).$$

(vii) *T is t_2 -transitive if and only if t satisfies the identity*

$$t(x, t(x, u, t(x, y, z)), z) = t(x, u, t(x, y, z)).$$

(viii) *T is R -symmetric if and only if t satisfies the identity*

$$t(t(x, y, z), x, z) = x.$$

(ix) *If t satisfies the identity*

$$t(t(x, y, z), x, z) = t(x, y, z)$$

then T is R -antisymmetric.

(x) *T is non-sharp if and only if t satisfies the identity*

$$t(x, x, y) = x.$$

(xi) *T is cyclically transitive if and only if t satisfies the identity*

$$t(x, t(x, y, t(x, z, u)), u) = t(x, y, t(x, z, u)).$$

Proof. Let $a, b, c, d, e \in A$.

(i) is clear.

(ii) t satisfies $t(z, t(x, y, z), x) = t(x, y, z)$ if and only if $(z, t(x, y, z), x) \in T$ for all $x, y, z \in A$.

“ \Rightarrow ”: $(a, t(a, b, c), c) \in T$ and hence $(c, t(a, b, c), a) \in T$.

“ \Leftarrow ”: If $(a, b, c) \in T$ then $(c, b, a) = (c, t(a, b, c), a) \in T$.

(iii) “ \Rightarrow ”: $(a, t(a, b, a), a) \in T$ and hence $t(a, b, a) = a$.

“ \Leftarrow ”: If $(a, b, a) \in T$ then $a = t(a, b, a) = b$.

(iv) t satisfies $t(t(x, y, z), z, x) = z$ if and only if $(t(x, y, z), z, x) \in T$ for all $x, y, z \in A$.

“ \Rightarrow ”: $(a, t(a, b, c), c) \in T$ and hence $(t(a, b, c), c, a) \in T$.

“ \Leftarrow ”: If $(a, b, c) \in T$ then $(b, c, a) = (t(a, b, c), c, a) \in T$.

(v) t satisfies $t(x, t(t(x, y, z), u, v), v) = t(t(x, y, z), u, v)$ if and only if $(x, t(t(x, y, z), u, v), v) \in T$ for all $x, y, z, u, v \in A$.

“ \Rightarrow ”: $(a, t(a, b, c), c), (t(a, b, c), t(t(a, b, c), d, e), e) \in T$ and hence $(a, t(t(a, b, c), d, e), e) \in T$.

“ \Leftarrow ”: If $(a, b, c), (b, d, e) \in T$ then $(a, d, e) = (a, t(t(a, b, c), d, e), e) \in T$.

(vi) t satisfies $t(t(x, u, t(x, y, z)), t(x, y, z), z) = t(x, y, z)$ if and only if $(t(x, u, t(x, y, z)), t(x, y, z), z) \in T$ for all $x, y, z, u \in A$.

“ \Rightarrow ”: $(a, t(a, b, c), c), (a, t(a, d, t(a, b, c)), t(a, b, c)) \in T$ and hence $(t(a, d, t(a, b, c)), t(a, b, c), c) \in T$.

“ \Leftarrow ”: If $(a, b, c), (a, d, b) \in T$ then $(d, b, c) = (t(a, d, t(a, b, c)), t(a, b, c), c) \in T$.

(vii) t satisfies $t(x, t(x, u, t(x, y, z)), z) = t(x, u, t(x, y, z))$ if and only if $(x, t(x, u, t(x, y, z)), z) \in T$ for all $x, y, z, u \in A$.

“ \Rightarrow ”: $(a, t(a, b, c), c), (a, t(a, d, t(a, b, c)), t(a, b, c)) \in T$ and hence $(a, t(a, d, t(a, b, c)), c) \in T$.

“ \Leftarrow ”: If $(a, b, c), (a, d, b) \in T$ then $(a, d, c) = (a, t(a, d, t(a, b, c)), c) \in T$.

(viii) t satisfies $t(t(x, y, z), x, z) = x$ if and only if $(t(x, y, z), x, z) \in T$ for all $x, y, z \in A$.

“ \Rightarrow ”: $(a, t(a, b, c), c) \in T$ and hence $(t(a, b, c), a, c) \in T$.

“ \Leftarrow ”: If $(a, b, c) \in T$ then $(b, a, c) = (t(a, b, c), a, c) \in T$.

(ix) If $(a, b, c), (b, a, c) \in T$ then $a = t(b, a, c) = t(t(a, b, c), a, c) = t(a, b, c) = b$.

(x) This is clear.

(xi) t satisfies $t(x, t(x, y, t(x, z, u)), u) = t(x, y, t(x, z, u))$ if and only if $(x, t(x, y, t(x, z, u)), u) \in T$ for all $x, y, z, u \in A$.

“ \Rightarrow ”: $(a, t(a, b, t(a, c, d)), t(a, c, d)), (a, t(a, c, d), d) \in T$ and hence $(a, t(a, b, t(a, c, d)), d) \in T$.

“ \Leftarrow ”: If $(a, b, c), (a, c, d) \in T$ then $t(a, b, d) = t(a, t(a, b, t(a, c, d)), d) = t(a, b, t(a, c, d)) = b$.

□

Lemma 2. *Let T be a ternary relation on A . Then*

$$|T| = \sum_{(a,b) \in A^2} |Z_T(a,b)|.$$

Proof.

$$T = \dot{\bigcup}_{(a,b) \in A^2} (\{a\} \times Z_T(a,b) \times \{b\}).$$

□

Corollary 1. *Let A be finite, $|A| = n$. If T is a centred ternary relation on A then $|T| \geq n^2$. Moreover, if T is centred then $|T| = n^2$ if and only if $|Z_T(x,y)| = 1$ for each $x, y \in A$.*

2. CONGRUENCES, HOMOMORPHISMS AND SUBSYSTEMS OF TERNARY RELATIONAL SYSTEMS

By a **ternary relational system** is meant a couple $\mathcal{T} = (A; T)$ where T is a ternary relation on A . \mathcal{T} is called **centred** if T is centred. As shown in the previous section, to every centred ternary relational system $\mathcal{T} = (A; T)$ there can be assigned an algebra $\mathcal{A}(T) = (A; t)$ with one ternary operation $t : A^3 \rightarrow A$ such that t is assigned to T . Now, we can introduce an inverse construction. It means that to every algebra $\mathcal{A} = (A; t)$ of type (3) there can be assigned a ternary relational system $\mathcal{T}(\mathcal{A}) = (A; T_t)$ where T_t is defined by

$$T_t := \{(x, y, z) \in A^3 \mid t(x, y, z) = y\}. \quad (2.1)$$

Of course, an assigned ternary relational system $\mathcal{T}(\mathcal{A}) = (A; T_t)$ need not be centred. However, if $\mathcal{T} = (A; T)$ is a centred ternary relational system and $\mathcal{A}(T) = (A; t)$ an assigned algebra then T_t is centred despite the fact that t is not determined uniquely. In fact, we have $(a, b, c) \in T_t$ if and only if $t(a, b, c) = b$ if and only if $(a, b, c) \in T$. Hence, we have proved the following

Lemma 3. *Let $\mathcal{T} = (A; T)$ be a centred ternary relational system, $\mathcal{A}(T) = (A; t)$ an assigned algebra and $\mathcal{T}(\mathcal{A}(T)) = (A; T_t)$ the ternary relational system assigned to $\mathcal{A}(T)$. Then $\mathcal{T}(\mathcal{A}(T)) = \mathcal{T}$.*

The best known correspondence between centred ternary relational systems and corresponding algebras of type (3) is the case of “betweenness”-relations and median algebras which was initiated by J. R. Isbell [5] and essentially developed by H.-J. Bandelt and J. Hedlíková [1]. However, there are also some essential differences between relational systems and the corresponding algebras. For binary relational systems it was described by the first and the third author in [2]. In what follows, we are going to handle it for the ternary case.

If $\mathcal{T} = (A; T)$ is a ternary relational system and E an equivalence relation on A then the **quotient relational system** \mathcal{T}/E is defined as the relational system $(A/E, T/E)$ where $T/E := \{([x]_E, [y]_E, [z]_E) \mid (x, y, z) \in T\}$. It is evident that E need not be a congruence on the assigned algebra $\mathcal{A}(T) = (A; t)$ and hence congruences on $\mathcal{T} = (A; T)$, respectively on $\mathcal{A}(T)$ are different concepts.

Similarly, by a **subsystem** of $\mathcal{T} = (A; T)$ is meant a couple of the form $(B, T|B)$ with a non-empty subset B of A and $T|B := T \cap B^3$. One can easily see that this need not be a subalgebra of $\mathcal{A}(T) = (A; t)$.

Finally, by a **homomorphism** of a ternary relational system $\mathcal{T} = (A; T)$ into a ternary relational system $\mathcal{S} = (B; S)$ is meant a mapping $h : A \rightarrow B$ satisfying

$$(a, b, c) \in T \implies (h(a), h(b), h(c)) \in S.$$

A homomorphism h is called **strong** if for each triple $(p, q, r) \in S$ there exists $(a, b, c) \in T$ such that $(h(a), h(b), h(c)) = (p, q, r)$.

Now, we define the following concept.

Definition 5. A **t -homomorphism** from a centred ternary relational system $\mathcal{T} = (A; T)$ to a ternary relational system $\mathcal{S} = (B; S)$ is a homomorphism from \mathcal{T} to \mathcal{S} such that there exists an algebra $(A; t)$ assigned to \mathcal{T} such that $a, b, c, a', b', c' \in A$ and $(h(a), h(b), h(c)) = (h(a'), h(b'), h(c'))$ together imply $h(t(a, b, c)) = h(t(a', b', c'))$.

Theorem 3. Let $\mathcal{T} = (A; T)$ and $\mathcal{S} = (B; S)$ be centred ternary relational systems and $\mathcal{A}(T) = (A; t)$ and $\mathcal{B}(S) = (B; s)$ assigned algebras. Then every homomorphism from $\mathcal{A}(T)$ to $\mathcal{B}(S)$ is a t -homomorphism from \mathcal{T} to \mathcal{S} .

Proof. Let $a, b, c, a', b', c' \in A$. If $(a, b, c) \in T$ then $t(a, b, c) = b$ and hence $s(h(a), h(b), h(c)) = h(t(a, b, c)) = h(b)$ showing $(h(a), h(b), h(c)) \in S$. Thus h is a homomorphism from \mathcal{T} to \mathcal{S} .

Moreover, if $(h(a), h(b), h(c)) = (h(a'), h(b'), h(c'))$ then

$$h(t(a, b, c)) = s(h(a), h(b), h(c)) = s(h(a'), h(b'), h(c')) = h(t(a', b', c')).$$

Hence h is a t -homomorphism from \mathcal{T} to \mathcal{S} . \square

The theorem just proved says that every homomorphism of assigned algebras is a t -homomorphism of the original relational systems. Now we can show under which conditions the converse assertion becomes true.

Theorem 4. Let $\mathcal{T} = (A; T)$ and $\mathcal{S} = (B; S)$ be centred ternary relational systems. Then for every strong t -homomorphism h from \mathcal{T} to \mathcal{S} with assigned algebra $\mathcal{A}(T) = (A; t)$ there exists an algebra $\mathcal{B}(S) = (B; s)$ assigned to \mathcal{S} such that h is a homomorphism from $\mathcal{A}(T)$ to $\mathcal{B}(S)$.

Proof. Let h be a strong t -homomorphism from \mathcal{T} to \mathcal{S} . By definition there exists an algebra $\mathcal{A}(T) = (A; t)$ assigned to \mathcal{T} such that for all $a, b, c, a', b', c' \in A$ with

$(h(a), h(b), h(c)) = (h(a'), h(b'), h(c'))$ it holds $h(t(a, b, c)) = h(t(a', b', c'))$. Define a ternary operation s on B as follows:

$$s(h(x), h(y), h(z)) := h(t(x, y, z))$$

for all $x, y, z \in A$. Since h is strong and a t -homomorphism, s is correctly defined. For $a, b, c \in A$, if $(h(a), h(b), h(c)) \in S$ then there exists $(d, e, f) \in T$ such that $(h(d), h(e), h(f)) = (h(a), h(b), h(c))$. Now

$$s(h(a), h(b), h(c)) = h(t(a, b, c)) = h(t(d, e, f)) = h(e) = h(b).$$

If $(h(a), h(b), h(c)) \notin S$ then $(a, b, c) \notin T$ since h is a homomorphism from \mathcal{T} to \mathcal{S} and hence $t(a, b, c) \in Z_T(a, c)$, i. e. $(a, t(a, b, c), c) \in T$. Thus $(h(a), h(t(a, b, c)), h(c)) \in S$, i. e.

$$(h(a), s(h(a), h(b), h(c)), h(c)) \in S$$

whence $s(h(a), h(b), h(c)) \in Z_S(h(a), h(c))$. This shows that $\mathcal{B}(S)$ is an algebra assigned to \mathcal{B} . It is easy to see that h is a homomorphism from $\mathcal{A}(T)$ to $\mathcal{B}(S)$. \square

We are going to get connections between t -homomorphisms of relational systems and congruences on the assigned algebras.

Theorem 5. *Let $\mathcal{T} = (A; T)$, $\mathcal{S} = (B; S)$ be centred ternary relational systems. Then the following hold:*

- (i) *If h is a strong t -homomorphism from \mathcal{T} to \mathcal{S} then there exists an algebra $\mathcal{A}(T) = (A; t)$ assigned to \mathcal{T} such that $\ker h \in \text{Con } \mathcal{A}(T)$.*
- (ii) *If $\mathcal{A}(T) = (A; t)$ is an algebra assigned to \mathcal{T} and $\theta \in \text{Con } \mathcal{A}(T)$ then the canonical mapping $h : A \rightarrow A/\theta$ is a strong t -homomorphism from \mathcal{T} onto \mathcal{T}/θ .*

Proof. (i) Let h be a strong t -homomorphism from \mathcal{T} to \mathcal{S} . By definition and Theorem 4, there exist assigned algebras $\mathcal{A}(T) = (A; t)$, respectively $\mathcal{B}(S) = (B; s)$ such that h is a homomorphism of $\mathcal{A}(T)$ to $\mathcal{B}(S)$ and hence $\ker h \in \text{Con } \mathcal{A}(T)$.

(ii) Let $\mathcal{A}(T) = (A; t)$ be an algebra assigned to \mathcal{T} , $\theta \in \text{Con } \mathcal{A}(T)$ and $h : A \rightarrow A/\theta$ denote the canonical mapping. By definition of T/θ , if $(a, b, c) \in T$ then $(h(a), h(b), h(c)) \in T/\theta$ and hence h is a homomorphism from \mathcal{T} to \mathcal{T}/θ . If, moreover, $a, b, c, a', b', c' \in A$ and $(h(a), h(b), h(c)) = (h(a'), h(b'), h(c'))$ then

$$h(t(a, b, c)) = t(h(a), h(b), h(c)) = t(h(a'), h(b'), h(c')) = h(t(a', b', c')).$$

Therefore h is a t -homomorphism from \mathcal{T} onto \mathcal{T}/θ . Obviously, h is strong. \square

Definition 6. Let $\mathcal{T} = (A; T)$ be a centred ternary relational system. An equivalence relation θ on A is called a **t -congruence** on \mathcal{T} if there exists an algebra $\mathcal{A}(T) = (A; t)$ assigned to \mathcal{T} such that $\theta \in \text{Con } \mathcal{A}(T)$. A subset B of A is called a **t -subsystem** of \mathcal{T} if there exists an algebra $\mathcal{A}(T) = (A; t)$ assigned to \mathcal{T} such that $(B; t)$ is a subalgebra of $\mathcal{A}(T)$.

Example 2. Consider $A = \{a, b, c, d\}$ and the ternary relation T on A defined as follows: $T := A \times \{d\} \times A$. Then $d \in Z_T(x, y)$ for each $x, y \in A$ and hence T is centred and its median is non-empty, in fact $M_T(x, y, z) = \{d\}$ for all $x, y, z \in A$. For $B = \{a, b, c\}$, $\mathcal{B} = (B; T|B)$ is a subsystem of $\mathcal{A} = (A; T)$ but it is not a t -subsystem. Namely, for every $x, y, z \in A$ t can be defined in the unique way as follows: $t(x, y, z) := d$. Hence, $(\{a, b, c\}; t)$ is not a subalgebra of $(A; t)$. On the contrary, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$ are t -subsystems of \mathcal{A} .

Remark 2. Let $\mathcal{A} = (A; t)$, $\mathcal{B} = (B; s)$ be algebras of type (3) and $h : A \rightarrow B$ a homomorphism from \mathcal{A} to \mathcal{B} . Put $\mathcal{T}(A) := (A; T_t)$ and $\mathcal{S}(B) := (B; S_s)$ where T_t , S_s are defined by (2.1). Then h need not be a t -homomorphism of $\mathcal{T}(A)$ to $\mathcal{S}(B)$, see the following example.

Example 3. Let $A = \{-1, 0, 1\}$, $B = \{1, 0\}$ and $t(x, y, z) = x \cdot y$, $s(x, y, z) = x \cdot y$, where “ \cdot ” is the multiplication of integers. Let $h : A \rightarrow B$ be defined by $h(x) = |x|$. Then h is clearly a homomorphism from $\mathcal{A} = (A; t)$ to $\mathcal{B} = (B; s)$ and

$$T_t = (A \times \{0\} \times A) \cup (\{1\} \times A^2).$$

There exists exactly one algebra $(A; t^*)$ assigned to $\mathcal{T}(A)$, namely where

$$t^*(x, y, z) := \begin{cases} y & \text{if } y = 0 \text{ or } x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Now $h(-1) = h(1)$ but $h(t^*(-1, -1, 1)) = h(0) = 0 \neq 1 = h(1) = h(t^*(1, 1, 1))$. Thus h is not a t -homomorphism.

We can prove the following:

Theorem 6. *If $\mathcal{A} = (A; t)$ and $\mathcal{B} = (B; s)$ are algebras of type (3), \mathcal{A} satisfies the identity*

$$t(x, t(x, y, z), z) = t(x, y, z)$$

and $\mathcal{T}(A) = (A; T_t)$ and $\mathcal{S}(B) = (B; S_s)$ denote the relational systems corresponding to \mathcal{A} and \mathcal{B} , respectively, as defined by (2.1) then every homomorphism h from \mathcal{A} to \mathcal{B} is a t -homomorphism from $\mathcal{T}(A)$ to $\mathcal{S}(B)$.

Proof. Let $a, b, c, d, e, f \in A$. If $(a, b, c) \in \mathcal{T}(A)$ then $t(a, b, c) = b$ and hence

$$s(h(a), h(b), h(c)) = h(t(a, b, c)) = h(b)$$

whence $(h(a), h(b), h(c)) \in \mathcal{S}(B)$. This shows that h is a homomorphism from $\mathcal{T}(A)$ to $\mathcal{S}(B)$. Obviously, t is assigned to T_t . Finally, $(h(a), h(b), h(c)) = (h(d), h(e), h(f))$ implies

$$h(t(a, b, c)) = s(h(a), h(b), h(c)) = s(h(d), h(e), h(f)) = h(t(d, e, f))$$

which shows that h is a t -homomorphism from $\mathcal{T}(A)$ to $\mathcal{S}(B)$. \square

3. DERIVED BINARY SYSTEMS

Let T be a ternary relation on A and p an arbitrary, but fixed element of A . Then

$$R_T := \{(x, y) \in A^2 \mid (x, y, p) \in T\}$$

is called the binary relation **p -derived from T** . Moreover, put $x \circ y := t(x, y, p)$ for all $x, y \in A$ if T is centred and t is an assigned operation.

If T is reflexive then R_T is reflexive, too. If, moreover, T is centred then Theorem 2 implies $x \circ x = x$, the idempotency of the operation \circ which is in accordance with (i) of Theorem 8 in [3].

Similarly, if T is R -symmetric then R_T is symmetric. If, moreover, T is centred then Theorem 2 implies $(x \circ y) \circ x = x$ which is identity (ii) of Theorem 8 in [3] characterizing symmetric binary relations (for directed relational systems).

If T is R -antisymmetric then R_T is antisymmetric. If, moreover, T is centred then Theorem 2 yields that $(x \circ y) \circ x = x \circ y$ which, if satisfied for all $p \in A$, is a sufficient condition for the antisymmetry of R_T . This condition is also a sufficient condition for the antisymmetry of binary relations (see (v) of Theorem 8 in [3]).

If T is R -transitive then R_T is transitive. If, moreover, T is centred then Theorem 2 implies $x \circ ((x \circ y) \circ u) = (x \circ y) \circ u$ which is just identity (iii) of Theorem 8 in [3] characterizing transitivity of binary relations.

Let us recall from [3] that a binary relation R on A is **(upward) directed** if

$$U_R(a, b) := \{x \in A \mid (a, x), (b, x) \in R\} \neq \emptyset \text{ for all } a, b \in A.$$

Although reflexivity, R -symmetry, R -antisymmetry and R -transitivity of a ternary relation T on A yields the corresponding property of R_T , we are not able to show that if T is centred then R_T is directed. However, our characterization of the corresponding properties for binary relations by means of the induced binary operations in [3] are possible for directed relations only.

Example 4. Put $A := \{x, y, z\}$ and

$$T := \{(x, z, y)\} \cup \{(a, y, b) \mid (a, b) \in A^2 \setminus \{(x, y)\}\}.$$

Then T is centred because $Z_T(x, y) = \{z\}$ and $Z_T(a, b) = \{y\}$ for $(a, b) \in A^2 \setminus \{(x, y)\}$. Put $p := y$ and consider the p -derived binary relation R_T on A . Then

$$x \circ (x \circ y) = t(x, t(x, y, y), y) = t(x, z, y) = z = t(x, y, y) = x \circ y,$$

but

$$y \circ (x \circ y) = t(y, t(x, y, y), y) = t(y, z, y) = y \neq z = t(x, y, y) = x \circ y.$$

Thus $y \circ (x \circ y) \neq (x \circ y)$. According to (ii) of Theorem 6 in [3], R_T is not directed.

Remark 3. Theorem 6 in [3] says that for a groupoid $(G; \circ)$ the following are equivalent:

- (i) There exists a directed relational system $(G; R)$ with a reflexive relation R such that $(G; \circ)$ corresponds to $(G; R)$.
- (ii) $(G; \circ)$ satisfies the identities $x \circ x = x$ and $x \circ (x \circ y) = y \circ (x \circ y) = x \circ y$.

We are going to show a sufficient condition for R_T to be directed.

Theorem 7. *Let T be a reflexive ternary relation on A such that $Z_T(a, c) \cap Z_T(b, c) \neq \emptyset$ for all $a, b, c \in A$. Let $p \in A$ and R_T denote the binary relation p -derived from T . Then R_T is directed.*

Proof. Due to the assumption, T is centred and hence we can consider a ternary operation t on A assigned to T such that $t(a, b, c) \in Z_T(a, c) \cap Z_T(b, c)$ if $(a, b, c) \in A^3 \setminus T$. Since T is reflexive, we have $x \circ x = t(x, x, p) = x$.

First assume $(x, y) \in R_T$. Then $(x, y, p) \in T$. Thus $t(x, y, p) = y$ and hence

$$x \circ (x \circ y) = t(x, t(x, y, p), p) = t(x, y, p) = x \circ y.$$

Since T is reflexive, we obtain

$$y \circ (x \circ y) = t(y, t(x, y, p), p) = t(y, y, p) = y = t(x, y, p) = x \circ y.$$

Now suppose $(x, y) \notin R_T$. Then

$$x \circ (x \circ y) = t(x, t(x, y, p), p) = t(x, y, p) = x \circ y.$$

Since $t(x, y, p) \in Z_T(x, p) \cap Z_T(y, p)$ we have also $t(x, y, p) \in Z_T(y, p)$ and hence

$$y \circ (x \circ y) = t(y, t(x, y, p), p) = t(x, y, p) = x \circ y.$$

We have shown that \circ satisfies (ii) of Theorem 6 in [3]. Thus R_T is directed. \square

The converse assertion is also true. For a binary relation R on A and a fixed element $p \in A$ we define

$$T_p(R) := \{(x, y, p) \mid (x, y) \in R\} \cup \{(x, x, y) \mid x, y \in A\}. \quad (3.1)$$

Then we can prove

Proposition 1. *Let R be a reflexive binary relation on A , $p \in A$ and $T_p(R)$ defined by (3.1). Then $T_p(R)$ is a centred ternary relation on A and its p -derived binary relation is just R .*

Proof. It is evident that $T_p(R)$ is a ternary relation on A , its p -derived binary relation is just R and $Z_{T_p(R)}(x, y) \supseteq \{y\} \neq \emptyset$ for all $x, y \in A$, i. e. $T_p(R)$ is centred. \square

In what follows, we focus on the relation between ternary relations preserving a given function and properties of assigned operations.

Definition 7. Let T be a ternary relation and f an m -ary operation on A . We say that f **preserves** T if

$$(a_1, b_1, c_1), \dots, (a_m, b_m, c_m) \in T \text{ implies } (f(a_1, \dots, a_m), f(b_1, \dots, b_m), f(c_1, \dots, c_m)) \in T.$$

It is worth noticing that the set of all operations on A preserving a given relation T forms a so-called clone. This topic is intensively investigated in contemporary algebra.

Definition 8. Let f be an m -ary and g an n -ary operation on A . We say that f and g **commute with each other** if

$$f(g(x_{11}, \dots, x_{1n}), \dots, g(x_{m1}, \dots, x_{mn})) = g(f(x_{11}, \dots, x_{m1}), \dots, f(x_{1n}, \dots, x_{mn}))$$

for all $x_{11}, \dots, x_{1n}, \dots, x_{m1}, \dots, x_{mn} \in A$.

We remark that also the set of all operations on A commuting with a given operation f forms a clone. Our next task is to compare both of these concepts.

Lemma 4. *If T is a centred ternary relation, f an m -ary operation on A commuting with a ternary operation t assigned to T then f preserves T .*

Proof. Let t be a ternary operation assigned to T . Assume $(a_1, b_1, c_1), \dots, (a_m, b_m, c_m) \in T$. Let f commute with t . Then $t(a_i, b_i, c_i) = b_i$ for $i = 1, \dots, m$ and hence

$$\begin{aligned} & t(f(a_1, \dots, a_m), f(b_1, \dots, b_m), f(c_1, \dots, c_m)) \\ &= f(t(a_1, b_1, c_1), \dots, t(a_m, b_m, c_m)) \\ &= f(b_1, \dots, b_m) \end{aligned}$$

showing $(f(a_1, \dots, a_m), f(b_1, \dots, b_m), f(c_1, \dots, c_m)) \in T$. □

Clearly the sufficient condition used in the previous Lemma is not necessary. Such a condition is as follows.

Theorem 8. *If T is a centred ternary relation, f an m -ary operation on A and t a ternary operation assigned to T then f preserves T if and only if it satisfies the following identity:*

$$\begin{aligned} & t(f(x_1, \dots, x_m), f(t(x_1, y_1, z_1), \dots, t(x_m, y_m, z_m)), f(z_1, \dots, z_m)) \\ &= f(t(x_1, y_1, z_1), \dots, t(x_m, y_m, z_m)). \end{aligned} \tag{3.2}$$

Proof. Assume that f preserves T . Since t is assigned to T we have $(x_i, t(x_i, y_i, z_i), z_i) \in T$ for all $i = 1, \dots, m$. Hence

$$(f(x_1, \dots, x_m), f(t(x_1, y_1, z_1), \dots, t(x_m, y_m, z_m)), f(z_1, \dots, z_m)) \in T.$$

Thus (3.2) holds.

Conversely, assume that f satisfies (3.2) and $(a_1, b_1, c_1), \dots, (a_m, b_m, c_m) \in T$. Then

$$t(a_i, b_i, c_i) = b_i$$

for $i = 1, \dots, m$, and hence

$$\begin{aligned} & t(f(a_1, \dots, a_m), f(b_1, \dots, b_m), f(c_1, \dots, c_m)) \\ &= t(f(a_1, \dots, a_m), f(t(a_1, b_1, c_1), \dots, t(a_m, b_m, c_m)), f(c_1, \dots, c_m)) \\ &= f(t(a_1, b_1, c_1), \dots, t(a_m, b_m, c_m)) \\ &= f(b_1, \dots, b_m) \end{aligned}$$

proving $(f(a_1, \dots, a_m), f(b_1, \dots, b_m), f(c_1, \dots, c_m)) \in T$. Hence, f preserves T . \square

4. MEDIAN-LIKE ALGEBRAS

The concept of a median algebra was introduced by J. R. Isbell (see [5]) as follows: An algebra $\mathcal{A} = (A; t)$ of type (3) is called a **median algebra** if it satisfies the following identities:

- (M1) $t(x, x, y) = x$;
- (M2) $t(x, y, z) = t(y, x, z) = t(y, z, x)$;
- (M3) $t(t(x, y, z), v, w) = t(x, t(y, v, w), t(z, v, w))$.

It is well-known (see e.g. [1], [5]) that the ternary relation T_t on A assigned to t via (2.1) is centred and, moreover, $|M_{T_t}(a, b, c)| = 1$ for all $a, b, c \in A$. In fact, $t(a, b, c) \in M_{T_t}(a, b, c)$. In particular, having a distributive lattice $\mathcal{L} = (L; \vee, \wedge)$ then $m(x, y, z) = M(x, y, z)$ and putting $t(x, y, z) := m(x, y, z)$, one obtains a median algebra. Conversely, every median algebra can be embedded into a distributive lattice. Moreover, the assigned ternary relation T_t is the so-called “betweenness”, see [10] and [11].

In what follows, we focus on the case when $M_T(a, b, c) \neq \emptyset$ for all $a, b, c \in A$ and $t(a, b, c) \in M_T(a, b, c)$ also in case $|M_T(a, b, c)| \geq 1$.

Definition 9. A **median-like algebra** is an algebra $(A; t)$ of type (3) where t satisfies (M1) and (M2) and where there exists a centred ternary relation T on A such that $t(x, y, z) \in M_T(x, y, z)$ for all $x, y, z \in A$.

Theorem 9. An algebra $\mathcal{A} = (A; t)$ of type (3) is median-like if t satisfies (M1), (M2) and

$$t(x, t(x, y, z), y) = t(y, t(x, y, z), z) = t(z, t(x, y, z), x) = t(x, y, z). \quad (4.1)$$

Proof. If $T := \{(x, y, z) \in A^3 \mid t(x, y, z) = y\}$ then $t(x, y, z) \in M_T(x, y, z)$ for all $x, y, z \in A$. \square

Lemma 5. Every median algebra is a median-like algebra.

Proof. As shown in [5], identities (M1), (M2), (M3) are equivalent to the identity

$$t(x, t(x, z, w), t(y, z, w)) = t(x, z, w).$$

Putting $w = y$ and using (M1) and (M2), we derive

$$t(x, t(x, z, y), y) = t(x, t(x, z, y), t(y, z, y)) = t(t(x, x, y), z, y) = t(x, z, y)$$

whence (4.1) follows since according to (M2) we have $t(u, v, w) = t(x, y, z)$ for any permutation (u, v, w) of (x, y, z) . \square

The following examples show that a median-like algebra need not be a median algebra.

Example 5. Put $A := \{1, 2, 3, 4, 5\}$, let t denote the ternary operation on A defined by $t(x, x, y) = t(x, y, x) = t(y, x, x) := x$ for all $x, y \in A$ and $t(x, y, z) := \min(x, y, z)$ for all $x, y, z \in A$ with $x \neq y \neq z \neq x$ and put $T := \{(x, x, y) \mid x, y \in A\} \cup \{(y, x, x) \mid x, y \in A\} \cup \{(x, y, z) \in A^3 \mid y < x < z\} \cup \{(x, y, z) \in A^3 \mid y < z < x\}$. Then t satisfies (M1) and (M2) and $t(x, y, z) \in M_T(x, y, z)$ for all $x, y, z \in A$. This shows that $(A; t)$ is median-like. However, this algebra is not a median algebra since

$$t(t(1, 3, 4), 2, 5) = t(1, 2, 5) = 1 \neq 2 = t(1, 2, 2) = t(1, t(3, 2, 5), t(4, 2, 5))$$

and hence (M3) is not satisfied.

Example 6. Consider the lattice M_3 given in FIGURE 1 below.

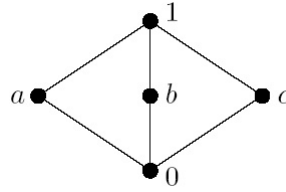


FIGURE 1.

Then M_3 is not distributive, $m(a, b, c) = 0$ and $M(a, b, c) = 1$. Define $(x, y, z) \in T$ if and only if $y \in [x \wedge z, x \vee z]$. Let t be an assigned operation defined as follows

$$t(x, y, z) := m(x, y, z).$$

Then $t(x, y, z) \in M_T(x, y, z)$ for all triples of elements x, y, z and hence $(M_3; t)$ is a median-like algebra. However, it is not a median algebra because identity (M3) is violated:

$$t(t(a, b, c), a, 1) = t(0, a, 1) = a \neq 1 = t(a, 1, 1) = t(a, t(b, a, 1), t(c, a, 1)).$$

The previous example motivated us to state a general construction for lattices which need not be neither distributive nor modular.

Theorem 10. Let $\mathcal{L} = (L; \vee, \wedge)$ be a lattice. Define $t_1(x, y, z) := m(x, y, z)$, $t_2(x, y, z) := M(x, y, z)$. Then $\mathcal{A}_1 := (L; t_1)$ and $\mathcal{A}_2 := (L; t_2)$ are median-like algebras. Moreover, the following conditions are equivalent

- (a) $\mathcal{A}_1 = \mathcal{A}_2$;
- (b) \mathcal{A}_1 is a median algebra;
- (c) \mathcal{L} is distributive.

Proof. Since both $m(x, y, z)$ and $M(x, y, z)$ satisfy (M1) and (M2) and $m(x, y, z), M(x, y, z) \in [m(x, y, z), M(x, y, z)] = M_T(x, y, z)$ for $(x, y, z) \in L^3$ and $T := \{(x, y, z) \in L^3 \mid x \wedge z \leq y \leq x \vee z\}$, $\mathcal{A}_1, \mathcal{A}_2$ are median-like algebras. It is well-known that $m(x, y, z) = M(x, y, z)$ if and only if \mathcal{L} is distributive which proves (a) \Leftrightarrow (c). The implication (c) \Rightarrow (b) is well-known (see e.g. [1], [5]). Finally, we prove (b) \Rightarrow (c). Assume that (b) holds but (c) does not. Then \mathcal{L} contains either $\mathcal{M}_3 = (\{0, a, b, c, 1\}; \vee, \wedge)$ or $\mathcal{N}_5 = (\{0, a, b, c, 1\}; \vee, \wedge)$ (with $a < c$) as a sublattice. In the first case we have

$$t(t(a, b, c), a, 1) = t(0, a, 1) = a \neq 1 = t(a, 1, 1) = t(a, t(b, a, 1), t(c, a, 1))$$

whereas in the second case

$$t(t(c, b, a), a, 1) = t(a, a, 1) = a \neq c = t(c, 1, a) = t(c, t(b, a, 1), t(a, a, 1))$$

which shows that (M3) does not hold. This is a contradiction to (b). Hence (c) holds. \square

Comparing our definition with Theorem 2, we conclude:

Corollary 2. An algebra $(A; t)$ of type (3) is median-like if t satisfies (M2) and if it is assigned to a centred antisymmetric or non-sharp ternary relation on A .

Let us mention that median-like algebras form a variety because they are defined by identities. Moreover, this variety is congruence distributive, i. e. $\text{Con}\mathcal{A}$ is distributive for every median-like algebra \mathcal{A} , because the operation t is a majority term, i. e. it satisfies by (M1) and (M2)

$$t(x, x, y) = t(x, y, x) = t(y, x, x) = x.$$

Theorem 11. let $\mathcal{L} = (L; \vee, \wedge)$ be a lattice and t a ternary operation on L satisfying (M1) and (M2) and $t(x, y, z) \in [m(x, y, z), M(x, y, z)]$ for all $x, y, z \in A$. Then $\mathcal{A} := (L; t)$ is a median-like algebra.

Proof. Put $T := \{(x, y, z) \in L^3 \mid x \wedge z \leq y \leq x \vee z\}$. Then $M_T(a, b, c) = [m(a, b, c), M(a, b, c)]$ for all $a, b, c \in L$. Hence $t(a, b, c) \in M_T(a, b, c)$ for all $a, b, c \in L$ showing that \mathcal{A} is a median-like algebra. \square

5. CYCLIC ALGEBRAS

Apart from the “betweenness” relation, another ternary relation plays an important role in mathematics. It is the so-called **cyclic order**, see e.g. [4], [9] and references there.

Definition 10. A ternary relation T on A is called **asymmetric** if

$$(a, b, c) \in T \text{ for } a \neq b \neq c \text{ implies } (c, b, a) \notin T. \quad (5.1)$$

A ternary relation C on A is called a **cyclic order** if it is cyclic, asymmetric, cyclically transitive and satisfies $(a, a, a) \in C$ for each $a \in A$.

Remark 4. Let C be a cyclic order on a set A . Then $(a, b, a) \notin C$ for all $a, b \in A$ with $a \neq b$. Namely, if $(a, b, a) \in C$ then, by (5.1), $(a, b, a) \notin C$, a contradiction. Since C is cyclic, we have also $(a, a, b), (b, a, a) \notin C$.

Applying (5.1), we derive immediately

Lemma 6. A centred ternary relation T on A is asymmetric if and only if any assigned ternary operation t satisfies the implication:

$$(t(x, y, z) = y \text{ and } x \neq y \neq z) \implies t(z, y, x) \neq y. \quad (5.2)$$

Similarly as for “betweenness” relations, we can derive an algebra of type (3) for a centred cyclic order by means of its assigned operation.

Definition 11. A **cyclic algebra** is an algebra assigned to a cyclic relation.

Cyclic algebras can be characterized by certain identities and the implication (5.2) as follows.

Theorem 12. An algebra $\mathcal{A} = (A; t)$ of type (3) is a cyclic algebra if and only if it satisfies (5.2) and

$$\begin{aligned} t(x, t(x, y, z), z) &= t(x, y, z), \\ t(t(x, y, z), z, x) &= z, \\ t(x, t(x, y, t(x, z, u)), u) &= t(x, y, t(x, z, u)), \\ t(x, x, x) &= x. \end{aligned}$$

Proof. Assume that $\mathcal{A} = (A; t)$ satisfies the above identities and (5.2). By Theorem 1 and the first identity, t is an assigned operation of a certain centred ternary relation C on A . By Theorem 2 and the second and third identity, C is cyclic and cyclically transitive. The fourth identity gets $(x, x, x) \in C$ for each $x \in A$. Finally, Lemma 6 yields that C is asymmetric and hence a cyclic order on A . Of course, t is an assigned operation of C and hence $\mathcal{A} = (A; t)$ is a cyclic algebra.

The converse follows directly by Definition 11. □

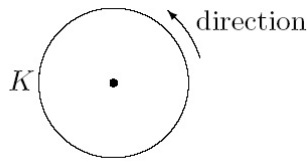


FIGURE 2.

Example 7. Let K be a circle in a plane with a given direction, see FIGURE 2. Define a ternary relation C on K as follows:

- $(a, a, a) \in C$ for each $a \in K$ and
- $(a, b, c) \in C$ if $a \rightarrow b$ and $b \rightarrow c$ for $a \neq b \neq c$.

It is an easy exercise to check that C is a cyclic order on K . If $a, b \in K$ then either $a = b$ and hence $Z_C(a, a) = \{a\}$ or $a \neq b$ thus $Z_C(a, b)$ equals the arc of K between a and b , i. e. it contains a continuum of points. Hence C is centred. For any assigned operation t , the algebra $\mathcal{A}(C) = (K; t)$ is a cyclic algebra.

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